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Deformed Minimal Models and Generalized Toda Theory

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ABSTRACT

We introduce a generalization of A_r -type Toda theory based on a non-abelian group G , which we call the (A_r, G) -Toda theory, and its affine extensions in terms of gauged Wess-Zumino-Witten actions with deformation terms. In particular, the affine $(A_1, SU(2))$ -Toda theory describes the integrable deformation of the minimal conformal theory for the critical Ising model by the operator $\Phi_{(2,1)}$. We derive infinite conserved charges and soliton solutions from the Lax pair of the affine $(A_1, SU(2))$ -Toda theory. Another type of integrable deformation which accounts for the $\Phi_{(3,1)}$ -deformation of the minimal model is also found in the gauged Wess-Zumino-Witten context and its infinite conserved charges are given.

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Recently, using the language of operator algebra, Zamolodchikov has shown that there exist some relevant perturbation around conformal field theory which preserve integrability.[1] In particular, when degenerate fields $\Phi_{(1,2)}, \Phi_{(2,1)}, \Phi_{(1,3)}$ and $\Phi_{(3,1)}$ are taken as the perturbations, he suggested that the resulting field theories may possess non-trivial integrals of motion and worked out explicitly for several examples. In the Lagrangian framework, there have been attempts to explain these particular perturbations in terms of the affine extension of Toda field theories[2][3]. In general, arbitrary coset conformal models can be formulated in terms of the gauged Wess-Zumino-Witten (WZW) action[4]. Using this fact and also generalizing the recent work of Bakas for the parafermion coset model[11], one of us (Q.P.) has recently shown that an integrable deformation of G/H-coset models is possible when the gauged WZW action for the G/H-coset model is added by a potential energy term $Tr(gTg^{-1}\bar{T})$, where algebra elements T, \bar{T} belong to the center of the algebra \mathfrak{h} associated with the subgroup H[6].

In this Letter, we consider two types of integrable deformations, by operators $\Phi_{(2,1)}$ and $\Phi_{(3,1)}$, of the minimal model corresponding to the coset $(SU(2)_N \times SU(2)_N)/SU(2)_{2N}$ where N denotes the level. This corresponds to the deformation of the critical Ising model for $N = 1$ and that of $c = 1$ theory in the super conformal minimal series for $N = 2$. We formulate the minimal model in terms of the gauged Wess-Zumino-Witten(WZW) action and show that the integrable deformations by $\Phi_{(2,1)}$ and $\Phi_{(3,1)}$ can be obtained by adding potential terms; $Tr(g_1^{-1}g_2 + g_2^{-1}g_1)$ and $Tr(g_1^{-1}L^a g_1 L^b)Tr(g_2^{-1}M^a g_2 M^b)$ respectively, where g_1, g_2 are Lie group $SU(2)$ -valued fields and $\{L^a\}, \{M^a\}$ are two sets of generators of the Lie algebra $su(2)$. In particular, the action for the $\Phi_{(2,1)}$ -deformation suggests a natural generalization of the abelian A_r -type Toda theory to the non-abelian (A_r, G) -Toda theory for a non-abelian group G and its affine extensions whereas the action for the $\Phi_{(2,1)}$ -deformation itself becomes the affine $(A_1, SU(2))$ -Toda theory. We demonstrate the integrability of both deformed models by deriving Lax pairs for them and also from which infinitely many conserved charges. We also derive n-soliton solutions for the affine $(A_1, SU(2))$ -Toda theory.

Recall that a lagrangian of the G/H-coset model is given in terms of the gauged WZW functional[4], which in light-cone variables is

$$S(g, A, \bar{A}) = S_{WZW}(g) + \frac{1}{2\pi} \int Tr(-A\bar{\partial}gg^{-1} + \bar{A}g^{-1}\partial g + Ag\bar{A}g^{-1} - A\bar{A}) \quad (1)$$

where $S_{WZW}(g)$ is the usual WZW action [5] for a map $g : M \rightarrow G$ on two-dimensional Minkowski space M . The connection A, \bar{A} gauge the anomaly free subgroup H of G . In this Letter, we take the diagonal embedding of H in $G_L \times G_R$, where G_L and G_R denote left and right group actions by multiplication ($g \rightarrow g_L g g_R^{-1}$), so that Eq.(1) becomes invariant under the vector gauge transformation ($g \rightarrow h g h^{-1}$ with $h : M \rightarrow H$). The restriction to the coset $(SU(2)_N \times SU(2)_N)/SU(2)_{2N}$ where N denotes the level of the Kac-Moody algebra is defined by the functional $\int [dg_1][dg_2][dA][d\bar{A}] \exp(iI_0(g_1, g_2, A, \bar{A}))$ where

$$I_0(g_1, g_2, A, \bar{A}) = NS_{WZW}(g_1, A, \bar{A}) + NS_{WZW}(g_2, A, \bar{A}) \quad (2)$$

where A and \bar{A} gauge simultaneously the diagonal subgroups of $SU(2) \times SU(2)$. The classical equations of motion for g_1 and g_2 arise in a form of zero curvature condition,

$$\begin{aligned} [\partial + g_1^{-1} \partial g_1 + g_1^{-1} A g_1, \bar{\partial} + \bar{A}] &= 0 \\ [\partial + g_2^{-1} \partial g_2 + g_2^{-1} A g_2, \bar{\partial} + \bar{A}] &= 0 \end{aligned} \quad (3)$$

whereas variations with respect to A and \bar{A} give the constraint equation,

$$\begin{aligned} -\bar{\partial} g_1 g_1^{-1} + g_1 \bar{A} g_1^{-1} - \bar{\partial} g_2 g_2^{-1} + g_2 \bar{A} g_2^{-1} - 2\bar{A} &= 0 \\ g_1^{-1} \partial g_1 + g_1^{-1} A g_1 + g_2^{-1} \partial g_2 + g_2^{-1} A g_2 - 2A &= 0. \end{aligned} \quad (4)$$

$\Phi_{(2,1)}$ -deformation and generalized Toda theory

Having introduced an action for the coset model, we now assert that an integrable deformation is possible when we add to the action a potential term in the following way:

$$I(g_1, g_2, A, \bar{A}, \kappa) = I_0(g_1, g_2, A, \bar{A}) - \frac{N\kappa}{2\pi} \int \text{Tr}(g_1^{-1} g_2 + g_2^{-1} g_1), \quad (5)$$

where κ is a coupling constant. The potential term transforms at the classical level as (doublet, singlet) in the convention of coset conformal field theory so that it corresponds to the operator $\Phi_{(2,1)}$. This changes Eq.(3) by

$$\begin{aligned} [\partial + g_1^{-1} \partial g_1 + g_1^{-1} A g_1, \bar{\partial} + \bar{A}] - \kappa(g_1^{-1} g_2 - g_2^{-1} g_1) &= 0 \\ [\partial + g_2^{-1} \partial g_2 + g_2^{-1} A g_2, \bar{\partial} + \bar{A}] + \kappa(g_1^{-1} g_2 - g_2^{-1} g_1) &= 0 \end{aligned} \quad (6)$$

while leaving the constraint equation unchanged. The main observation in proving the integrability of the model is that Eq.(6) is precisely the integrability condition of the linear 4×4 matrix equations with a spectral parameter λ ,

$$L_1(\lambda)\Psi \equiv [\partial + U_0 - \lambda T]\Psi = 0 \quad ; \quad L_2(\lambda)\Psi \equiv (\bar{\partial} + \bar{\mathcal{A}} + \frac{1}{\lambda}V_1)\Psi = 0 \quad (7)$$

where

$$U_0 = \mathcal{G}^{-1}\partial\mathcal{G} + \mathcal{G}^{-1}\mathcal{A}\mathcal{G} \quad , \quad V_1 = \mathcal{G}^{-1}\bar{T}\mathcal{G} \quad , \quad T = i\kappa\Sigma \quad , \quad \bar{T} = i\Sigma \quad (8)$$

and

$$\mathcal{G} \equiv \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad , \quad \mathcal{A} \equiv \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad , \quad \bar{\mathcal{A}} \equiv \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{pmatrix} \quad , \quad \Sigma \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (9)$$

with each entries being 2×2 matrices. Note that Eqs.(6) and (7) are non-abelian generalizations of the sine-Gordon equation and its linear equations. This may be seen easily if we take an $U(1)$ -reduction by setting $g_1 = g_2^{-1} = \exp(i\phi\sigma_3)$ and $A = \bar{A} = 0$ which satisfies the constraint equation trivially. We may generalize further to the non-abelian A_r -type Toda (affine Toda) equations if we take $\mathcal{G} = \text{diag}(g_1, \dots, g_r)$ for each g_i valued in a Lie group G , $\mathcal{A} = \text{diag}(A, \dots, A)$, $\bar{\mathcal{A}} = \text{diag}(\bar{A}, \dots, \bar{A})$ and $T = i\kappa 1 \otimes \Lambda$, $\bar{T} = i1 \otimes \bar{\Lambda}$ where Λ is the sum of simple roots of $sl(r)$ (the sum of simple roots minus the highest root for the affine Toda case) and $\bar{\Lambda} = \Lambda^t$. This, with the obvious generalization of the constraint equation, may be named as “the (A_r, G) -Toda (affine Toda) equation” so that Eq.(6) becomes the affine $(A_1, SU(2))$ -Toda equation. This type of non-abelian generalization of the Toda equation has been first considered by Mikhailov[7] but without the constraint equation.³ In fact, the constraint equation imposes a highly non-trivial restriction to the model. For example, without the constraint the abelian limit does not become the sine-Gordon equation but a pair of coupled two scalar field equations. Also, it is important to note that the constraint induces the gauge symmetry which allows different parameterizations of fields depending on particular choices of gauge fixing. To understand this more clearly, we note that the potential term added in Eq.(5) is invariant under the vector gauge transformation; $g_1 \rightarrow hg_1h^{-1}$, $g_2 \rightarrow hg_2h^{-1}$ for $h : M \rightarrow SU(2)$. It is easy to see that Eqs.(4) and (6) require A, \bar{A} to be flat, i.e.

³A different type of non-abelian Toda equation has also been considered in [10].

$[\partial + A, \bar{\partial} + \bar{A}] = 0$ which reflects the vector gauge invariance of the action. For the rest of the paper, we fix the gauge by setting $A = \bar{A} = 0$. In this gauge, the affine $(A_1, SU(2))$ -Toda equation takes a particularly simple form while the constraint can not be solved locally.⁴

$$\bar{\partial}(g_1^{-1}\partial g_1) = \kappa(g_2^{-1}g_1 - g_1^{-1}g_2) \quad ; \quad g_1^{-1}\partial g_1 + g_2^{-1}\partial g_2 = 0 . \quad (10)$$

With the linear equation as in Eq.(7), it is now more or less straightforward to obtain infinite conserved currents of the model. Define $\Phi = \Psi \exp(\lambda Tz) = \sum_{m=0}^{\infty} \lambda^{-m} \Phi_m$ with $\Phi_0 = 1$. If we parametrize Φ_m by

$$\Phi_{2m} \equiv \begin{pmatrix} p_{2m} & 0 \\ 0 & s_{2m} \end{pmatrix} , \quad \Phi_{2m+1} \equiv \begin{pmatrix} 0 & p_{2m+1} \\ s_{2m+1} & 0 \end{pmatrix} ; \quad m \geq 0 , \quad (11)$$

the linear equation in each order in λ changes into

$$\begin{aligned} \bar{\partial}p_{m+1} + ig_1^{-1}g_2s_m &= 0 \quad ; \quad \bar{\partial}s_{m+1} + ig_2^{-1}g_1p_m = 0 \\ \partial p_m + g_1^{-1}\partial g_1p_m &= i\kappa(p_{m+1} - s_{m+1}) \quad ; \quad \partial s_m + g_2^{-1}\partial g_2s_m = i\kappa(s_{m+1} - p_{m+1}) . \end{aligned} \quad (12)$$

This may be solved iteratively with an initial condition, $p_0 = s_0 = 1$. For example,

$$\begin{aligned} p_1 &= \frac{1}{2i\kappa}g_1^{-1}\partial g_1 - \frac{1}{2i\kappa} \int dz (g_1^{-1}\partial g_1)^2 - \frac{i}{2} \int d\bar{z} (g_1^{-1}g_2 + g_2^{-1}g_1) \\ s_1 &= -\frac{1}{2i\kappa}g_1^{-1}\partial g_1 - \frac{1}{2i\kappa} \int dz (g_1^{-1}\partial g_1)^2 - \frac{i}{2} \int d\bar{z} (g_1^{-1}g_2 + g_2^{-1}g_1) . \end{aligned} \quad (13)$$

The consistency condition; $\partial\bar{\partial}p_m = \bar{\partial}\partial p_m$ and $\partial\bar{\partial}s_m = \bar{\partial}\partial s_m$ in Eq.(12) gives rise to two sets of infinite conserved currents; $\bar{\partial}J_m^{(1)} + \partial\bar{J}_{m+2}^{(1)} = 0$; $m \geq 0$, and $\bar{\partial}J_m^{(2)} + \partial\bar{J}_{m+2}^{(2)} = 0$ where

$$\begin{aligned} \bar{J}_m^{(1)} &= ig_1^{-1}g_2s_m ; \quad J_{m+2}^{(1)} = \partial p_{m+1} = -g_1^{-1}\partial g_1p_{m+1} + i\kappa(s_{m+1} - p_{m+1}) \\ \bar{J}_m^{(2)} &= ig_2^{-1}g_1p_m ; \quad J_{m+2}^{(2)} = \partial s_{m+1} = g_1^{-1}\partial g_1s_{m+1} + i\kappa(p_{m+1} - s_{m+1}) . \end{aligned} \quad (14)$$

In particular, the energy-momentum conservation; $\bar{\partial}T_{\pm} + \partial\Theta_{\pm} = 0$, is given by

$$\begin{aligned} T_+ &= i\kappa(J_2^{(1)} + J_2^{(2)}) = (g_1^{-1}\partial g_1)^2 \\ \Theta_+ &= i\kappa(\bar{J}_0^{(1)} + \bar{J}_0^{(2)}) = -\kappa(g_1^{-1}g_2 + g_2^{-1}g_1) \end{aligned} \quad (15)$$

⁴In fact, there exists a different gauge choice which allows solutions of the constraint equation in terms of local fields. This case will be considered elsewhere.

while the other half of the conservation gives

$$\begin{aligned} T_- &= i\kappa(J_2^{(1)} - J_2^{(2)}) = \partial(g_1^{-1}\partial g_1) \\ \Theta_- &= i\kappa(\bar{J}_0^{(1)} - \bar{J}_0^{(2)}) = -\kappa(g_1^{-1}g_2 - g_2^{-1}g_1) \end{aligned} \quad (16)$$

It is interesting to observe that T_- in the abelian limit becomes $T_- = \partial^2\phi$ which is precisely the term added to improve the energy-momentum tensor in the Feigin-Fuchs construction.

Next, we derive n -soliton solutions by using the technique of Riemann problem with zeros[8]. Take a trivial solution of Eqs.(6) and (7) by $g_1 = g_2 = 1$ and $\Psi = \Psi^o = \exp(\lambda Tz - \lambda^{-1}\bar{T}\bar{z})$. Then, non-trivial solutions can be obtained if we “dress” Ψ_0 by $\Psi \equiv \Phi\Psi^o$, where Φ and Φ^{-1} are matrix functions each possessing n simple poles with the normalization $\Phi(z, \bar{z}, \lambda = \infty) = 1$. Here, we assume that all poles are distinct and consider soliton solutions for the group $U(2)$ only.⁵ The analytic property of Φ then leads to the linear equation for Ψ ,

$$(\partial + U_0 - \lambda T)\Psi = 0, \quad (\bar{\partial} + \frac{1}{\lambda}V_1)\Psi = 0 \quad (17)$$

where U_0 and V_1 , given by

$$U_0 \equiv -\partial\Phi\Phi^{-1} - \Phi\lambda T\Phi^{-1} + \lambda T; \quad V_1 \equiv -\lambda\bar{\partial}\Phi\Phi^{-1} + \Phi\bar{T}\Phi^{-1}, \quad (18)$$

are independent of λ . If we identify U_0 and V_1 with those of Eqs.(7)-(9) in the gauge $A = \bar{A} = 0$, we obtain precisely n -soliton solutions. In practice, these identifications can be simplified a lot if we make the following reductions;

i) \mathbf{Z}_2 -reduction

Since the diagonal structure of U_0 implies $Q^{-1}L_{1,2}(\lambda)Q = L_{1,2}(-\lambda)$ for the linear operators in Eq.(7) with $Q = \text{diag}(1, -1)$, Ψ may be reduced by the \mathbf{Z}_2 -action; $\Psi(\lambda) = Q\Psi(-\lambda)Q^{-1}$ or, $\Phi(\lambda) = Q\Phi(-\lambda)Q^{-1}$.

ii) Unitarity reduction

Unitary of U requires $\Psi^\dagger(\lambda) = \Psi^{-1}(\lambda^*)$ or $\Phi^\dagger(\lambda) = \Phi^{-1}(\lambda^*)$.

With reductions imposed, Φ and Φ^{-1} may be written as

$$\Phi = 1 + \sum_{s=1}^n \left(\frac{A^s}{\lambda - \nu_s} - \frac{QA^sQ^{-1}}{\lambda + \nu_s} \right), \quad \Phi^{-1} = 1 + \sum_{s=1}^n \left(\frac{B^s}{\lambda - \mu_s} - \frac{QB^sQ^{-1}}{\lambda + \mu_s} \right) \quad (19)$$

⁵ For real groups, the requirement of the distinct pole-assumption should be relaxed. The group $SU(2)$ and other Lie group cases as well as the study of properties of solitons will appear in a separate publication[9].

where $\mu_s = \nu_s^*$, and the matrix functions $A^s(z, \bar{z}) = B^{s\dagger}(z, \bar{z})$ are to be determined. Further determination on A^s and B^s comes through the evaluation of residues at $\lambda = \pm\nu_s, \pm\mu_s$ of the equation $\Phi\Phi^{-1} = 1$ as well as Eq.(18) which gives rise to

$$A^s + \sum_{\beta=1}^n A^s \left(\frac{B^\beta}{\nu_s - \mu_\beta} - \frac{QB^\beta Q^{-1}}{\nu_s + \mu_\beta} \right) = 0, \quad B^s + \sum_{\beta=1}^n \left(\frac{A^\beta}{\mu_s - \nu_\beta} - \frac{QA^\beta Q^{-1}}{\mu_s + \nu_\beta} \right) B^s = 0. \quad (20)$$

and

$$A^s D_{1,2}(\nu_s) \left[1 + \sum_{\alpha=1}^n \left(\frac{B^\alpha}{\nu_s - \mu_\alpha} - \frac{QB^\alpha Q^{-1}}{\nu_s + \mu_\alpha} \right) \right] = 0, \quad \left[1 + \sum_{\alpha=1}^n \left(\frac{A^\alpha}{\mu_s - \nu_\alpha} - \frac{QA^\alpha Q^{-1}}{\mu_s + \nu_\alpha} \right) \right] D_{1,2}(\mu_s) B^s = 0 \quad (21)$$

where $D_1 = \partial - \lambda T$, $D_2 = \bar{\partial} + \lambda^{-1} \bar{T}$. With ansätze $A_{ij}^\alpha = s_\alpha^i t_\alpha^j$, $B_{ij}^\alpha = n_\alpha^i m_\alpha^j$, it is easy to show that Eqs.(20) and (21) can be solved in terms of arbitrary constant vectors \bar{n}_α and $\bar{t}_\alpha = \bar{n}_\alpha^\dagger$,

$$n_\alpha^i = [\Psi^0(\mu_\alpha)]^{ij} \bar{n}_\alpha^j, \quad t_\alpha^i = [\Psi^o(\nu_\alpha)^{-1}]^{ji} \bar{t}_\alpha^j, \quad (22)$$

while m_α and s_α can be expressed in terms of t_α and n_α by

$$m_\alpha^l = -\frac{1}{2} \sum_{\beta=1}^n (w^l)_{\alpha\beta}^{-1} (\nu_\beta t_\beta^l), \quad s_\alpha^j = \frac{1}{2} \sum_{\beta=1}^n \nu_\alpha n_\beta^j (w^{j-1})_{\beta\alpha}^{-1}, \quad (23)$$

where $w_{\alpha\beta}^j \equiv \sum_{l=1}^2 \tau_{\alpha\beta}^{1+(j-l) \bmod 2} t_\alpha^l n_\beta^l$ and

$$\tau_{\alpha\beta}^n \equiv \frac{\nu_\alpha}{2} \left(\frac{1}{\nu_\alpha - \mu_\beta} + \frac{(-1)^n}{\nu_\alpha + \mu_\beta} \right). \quad (24)$$

Having determined Φ and Φ^{-1} as above, we finally obtain n -soliton solutions by evaluating Eq.(18) at $\lambda = 0$ such that

$$U_0 = -\partial\Phi\Phi^{-1}, \quad V_1 = \Phi\bar{T}\Phi^{-1}, \quad (25)$$

which gives $\Phi^{-1}(\lambda = 0) = \mathcal{G} = \text{diag}(g_1, g_2)$ in the gauge $A = \bar{A} = 0$. The result is

$$g_1 = 1 + \sum_{\alpha=1}^n \sum_{\beta=1}^n n_\alpha^1 (w^1)_{\alpha\beta}^{-1} \left(\frac{\nu_\beta}{\nu_\alpha^*} \right) t_\beta^1; \quad g_2 = 1 + \sum_{\alpha=1}^n \sum_{\beta=1}^n n_\alpha^2 (w^2)_{\alpha\beta}^{-1} \left(\frac{\nu_\beta}{\nu_\alpha^*} \right) t_\beta^2. \quad (26)$$

If n_α^1 is invertible, the n -soliton solution can be written more compactly with $N_\alpha \equiv n_\alpha^2 (n_\alpha^1)^{-1}$,

$$g_1 = 1 + \sum_{\alpha,\beta=1}^n \frac{\nu_\beta^2 - \nu_\alpha^{*2}}{\nu_\alpha^{*2}} \left[1 + \frac{\nu_\beta}{\nu_\alpha^*} N_\beta^\dagger N_\alpha \right]^{-1}; \quad g_2 = 1 + \sum_{\alpha,\beta=1}^n \frac{\nu_\beta^2 - \nu_\alpha^{*2}}{\nu_\alpha^{*2}} \left[1 + \frac{\nu_\beta}{\nu_\alpha^*} N_\beta^{\dagger-1} N_\alpha^{-1} \right]^{-1} \quad (27)$$

where

$$N_\alpha = [i \sin \theta \bar{n}_\alpha^1 + \cos \theta \bar{n}_\alpha^2] \times [\cos \theta \bar{n}_\alpha^1 + i \sin \theta \bar{n}_\alpha^2]^{-1} ; \quad \theta \equiv \nu_\alpha^* \kappa z - \frac{1}{\nu_\alpha^*} \bar{z} . \quad (28)$$

It is interesting to check that our soliton solution satisfies the constraint in Eq.(10). For n -soliton solutions,

$$g_1^{-1} \partial g_1 = -2i\kappa \sum_{\alpha=1}^n s_\alpha^1 t_\alpha^2 - 2i\kappa \sum_{\alpha=1}^n n_\alpha^2 m_\alpha^1 ; \quad g_2^{-1} \partial g_2 = -2i\kappa \sum_{\alpha=1}^n s_\alpha^2 t_\alpha^1 - 2i\kappa \sum_{\alpha=1}^n n_\alpha^1 m_\alpha^2. \quad (29)$$

With the help of the relation, $s_\alpha^1 t_\alpha^2 = \frac{1}{2} \sum_{\alpha,\beta=1}^n (\nu_\alpha n_\beta^1 (W^0)_{\beta\alpha}^{-1} t_\alpha^2) = -n_\alpha^1 m_a^2$, $s_\alpha^2 t_\alpha^1 = -n_\alpha^2 m_a^1$, it is straightfoward to check that n -soliton solution indeed satisfies the constraint.

$\Phi_{(3,1)}$ -deformation

Consider a deformation of the coset model Eq.(2) by adding a potential term;

$$I(g_1, g_2, A, \bar{A}, \beta) = I_0(g_1, g_2, A, \bar{A}) + \frac{\beta}{2\pi} \int \text{Tr}(g_1^{-1} L^a g_1 L^b) \text{Tr}(g_2^{-1} M^a g_2 M^b), \quad (30)$$

where β is a coupling constant and the summation is assumed for the repeated indices a and b . We also write $\{L^a = \sigma^a/2\}$ and $\{M^a = \sigma^a/2\}$ (σ^a are Pauli matrices) for two sets of generators of $SU(2)$ each corresponding to g_1 and g_2 respectively. The potential term in this case transforms as (triplet, singlet) so that it corresponds to the $\Phi_{(3,1)}$ -deformation. The equations of motion for g_1 and g_2 are

$$[\partial + g_1^{-1} \partial g_1 + g_1^{-1} A g_1, \bar{\partial} + \bar{A}] + \frac{\beta}{2} \text{Tr}(g_2^{-1} M^b g_2 M^a) [L^a, g_1^{-1} L^b g_1] = 0, \quad (31)$$

$$[\partial + g_2^{-1} \partial g_2 + g_2^{-1} A g_2, \bar{\partial} + \bar{A}] + \frac{\beta}{2} \text{Tr}(g_1^{-1} L^b g_1 L^a) [M^a, g_2^{-1} M^b g_2] = 0, \quad (32)$$

together with the constraint equations,

$$\begin{aligned} \text{Tr}(L^a (\bar{\partial} g_1 g_1^{-1} - g_1 \bar{A} g_1^{-1} + \bar{A})) + \text{Tr}(M^a (\bar{\partial} g_2 g_2^{-1} - g_2 \bar{A} g_2^{-1} + \bar{A})) &= 0, \\ \text{Tr}(L^a (-g_1^{-1} \partial g_1 - g_1^{-1} A g_1 + A)) + \text{Tr}(M^a (-g_2^{-1} \partial g_2 - g_2^{-1} A g_2 + A)) &= 0. \end{aligned} \quad (33)$$

In order to prove the integrability of the equation, here, unlike the previous case where we have embedded the product group $SU(2) \times SU(2)$ into the 4×4 matrix group, we work directly with the product group and denote $g : M \rightarrow SU(2) \times SU(2)$ by $g = g_1 g_2$.

Note that in these notations, the potential term takes a simple form; $\frac{\beta}{2\pi} \int \text{Tr} g^{-1} T g T$ with $T = L^a M^a$. The connections A, \bar{A} , which gauge both g_1 and g_2 , are given by $A = A^a L^a 1_2 + 1_1 A^a M^a$, $\bar{A} = \bar{A}^a L^a 1_2 + 1_1 \bar{A}^a M^a$ where $1_1, 1_2$ denote identity elements. The key observation for the integrability of this product-type model is that the equations of motion can be merged into a single zero curvature form with a spectral parameter λ such that

$$[\partial + g^{-1} \partial g + g^{-1} A g + \beta \lambda T, \bar{\partial} + \bar{A} + \frac{1}{\lambda} g^{-1} T g] = 0. \quad (34)$$

Using the identity $L^a L^b = \frac{1}{4} \delta^{ab} + \frac{i}{2} \epsilon^{abc} L^c$ for $SU(2)$, the equivalence between Eq.(34) and Eqs.(31) and (32) can be shown directly by evaluating the coefficient terms of $L^a 1_2$ and $1_1 M^a$ in Eq.(34) which gives rise to Eq.(31) and Eq.(32) whereas the coefficient term of $L^a M^b$ vanishes identically.

The zero curvature expression for the model again leads to the infinite conserved currents through the same procedure as in the generalized Toda case. By making use of the constraint equation, we first define B field by

$$g^{-1} \partial g + g^{-1} A g = A^a (L^a 1_2 + 1_1 M^a) + B^a (L^a 1_2 - 1_1 M^a) \quad (35)$$

and reexpress the linear equation in terms of $\Phi \equiv \Psi H^{-1} \exp(\lambda \beta T z) \equiv \sum_{m=0}^{\infty} \lambda^{-m} \Phi_m$, $\Phi_0 = 1$ where H solves the flat connection resulting from Eqs.(31)-(33); $A = H \partial H^{-1}$, $\bar{A} = H \bar{\partial} H^{-1}$,

$$\partial \Phi_m + A^a [L^a 1_2 + 1_1 M^a, \Phi_m] + B^a (L^a 1_2 - 1_1 M^a) \Phi_m + \beta [L^b M^b, \Phi_{m+1}] = 0, \quad (36)$$

$$\bar{\partial} \Phi_m + \bar{A}^a [L^a 1_2 + 1_1 M^a, \Phi_m] + (g_1^{-1} L^b g_1) (g_2^{-1} M^b g_2) \Phi_{m-1} = 0. \quad (37)$$

This may be solved iteratively with a parametrization of Φ_m by

$$\Phi_m \equiv \alpha_m 1_1 1_2 + \beta_m^a (L^a 1_2 + 1_1 M^a) + \gamma_m^a (L^a 1_2 - 1_1 M^a) + \delta_m^a \epsilon^{abc} L^b M^c + \eta_m^{ab} (L^a M^b + L^b M^a) \quad (38)$$

where $\eta_m^{ab} = \eta_m^{ba}$. In the gauge $A^a = \bar{A}^a = 0$, the iterative solutions for each component are;

$$\begin{aligned} \gamma_m^a &= \frac{i}{2\beta} (\partial \delta_{m-1}^a + i \epsilon^{abc} B^b \beta_{m-1}^c), \\ \delta_m^a &= -\frac{2i}{\beta} (\partial \gamma_{m-1}^a + B^a \alpha_{m-1} + i \epsilon^{abc} B^b \beta_{m-1}^c - \frac{1}{2} B^b \eta_{m-1}^{ab}) \\ \alpha_m &= -\frac{1}{2} \int B^a \gamma_m^a dz - \frac{1}{16} \int (\epsilon^{abc} \delta_{m-1}^a D_{(A)}^{bc} + 2 D_{(S)}^{ab} \eta_{m-1}^{ab}) d\bar{z} \end{aligned}$$

$$\begin{aligned}
\beta_m^a &= -\epsilon^{abc} \int \left(\frac{i}{2} B^b \gamma_m^c - \frac{1}{4} B^b \delta_m^c \right) dz - \frac{1}{4} \int (D_{(S)}^{ab} \beta_{m-1}^b + D_{(A)}^{ab} \gamma_{m-1}^b + \frac{i}{2} D_{(A)}^{ab} \delta_{m-1}^b) d\bar{z} \\
\eta_m^{ab} &= \frac{1}{2} \int (B^a \gamma_m^b + \frac{i}{2} B^a \delta_m^b - \frac{i}{2} B \cdot \delta_m \delta^{ab}) dz - \frac{1}{4} \int (D_{(S)}^{ab} \alpha_{m-1} + i \epsilon^{acd} D_{(S)}^{bc} \beta_{m-1}^d \\
&\quad + i \epsilon^{acd} D_{(A)}^{bc} \gamma_{m-1}^d + \frac{1}{4} \epsilon^{acd} D_{(A)}^{cd} \delta_{m-1}^b - \frac{1}{2} D_{(S)}^{cd} \epsilon^{ace} \epsilon^{bdf} \eta_{m-1}^{ef}) d\bar{z} + (a \Leftrightarrow b)
\end{aligned} \tag{39}$$

where

$$2D_{(S)}^{ab} \equiv D^{ab}(g_2^{-1}g_1) + D^{ba}(g_2^{-1}g_1), D^{ab}(g_1) = 2\text{Tr}(g_1^{-1}L^a g_1 L^b), 2D_{(A)}^{ab} \equiv D^{ab}(g_2^{-1}g_1) - D^{ba}(g_2^{-1}g_1). \tag{40}$$

With an initial condition $\Phi_0 = 1$, the first iterative solution, for example, is

$$\alpha_1 = \beta_1^a = \gamma_1^a = 0, \delta_1^a = -\frac{2i}{\beta} B^a, \eta_1^{ab} = \int (B^a B^b + \delta^{ab} B \cdot B) dz - \frac{1}{2} \int D_{(S)}^{ab} d\bar{z}. \tag{41}$$

Also, the consistency condition; $\partial\bar{\partial}\Phi_m = \bar{\partial}\partial\Phi_m$ yields a large set of infinite conservation laws in components through Eqs.(36)-(39) whose explicit form can be easily written down. Here, however we simply report the first non-trivial conservation law,

$$\beta\partial D_{(S)}^{ab} + 2\bar{\partial}(B^a B^b - B \cdot B \delta^{ab}) = 0 \tag{42}$$

which implies the energy-momentum conservation,

$$\beta\partial D_{(S)}^{aa} - 4\bar{\partial}(B \cdot B) = 0 \tag{43}$$

with $B^a = (g_1^{-1}\partial g_1)^a = -(g_2^{-1}\partial g_2)^a$.

In this Letter, we have introduced integrable deformations of coset models by adding potential terms which correspond to $\Phi_{(2,1)}$ and $\Phi_{(3,1)}$ operators. We have demonstrated the integrability by deriving explicitly infinite conservation laws in each case. The deformation by the operator $\Phi_{(3,1)}$ has not been taken seriously in previous works since the operator becomes irrelevant for $c < 1$. However, for $c = 1$, it becomes marginal and since our work proves the classical integrability of the deformed coset model with coset $(SU(2)_N \times SU(2)_N)/SU(2)_{2N}$ for all N , it would be interesting to understand physical implications of this type of deformation as well as those of the Toda-type deformation.

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